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ALGEBRA II

POLYNOMIAL BASICS

Polynomials are expressions obtained by adding, subtracting, and multiplying real numbers (**constants**) and one or several **variables**.

- Expressions connected by $+$ or $-$ signs are called **terms**. **Ex:** The polynomial $2x^3y - 7x$ has two terms.
- The **coefficient** of a term is the real number (non-variable) part.
- Two terms are sometimes called **like terms** if the power of each variable in the terms is the same. **Ex:** $7y^6x$ and $yxy^5 = xy^6$ are like terms. $2x^8$ and $16xy^7$ are not. Like terms can be added or subtracted into a single term.
- The **degree of a term** is the sum of the powers of each variable in the term. **Ex:** $2x^8$ and $16xy^6z$ both have degree eight. The degree of a nonzero constant is 0. The degree of the constant 0 is undefined (or taken to be $-\infty$).
- The **degree of a polynomial** is the highest degree of any of its terms.
- In a polynomial in one variable, the term with the highest degree is called the **leading term,** and its coefficient is the **leading coefficient.**

POLYNOMIALS IN ONE VARIABLE

- A polynomial of degree 2 is called **quadratic;** a polynomial of degree 3 is **cubic;** a polynomial of degree 4 is **quartic.**
- A **root,** or **zero,** of a polynomial is a (real) number such that when it is plugged into the variable, the polynomial evaluates to 0. **Ex:** $-\frac{1}{2}$ is a root of the polynomial $2x^2 - 5x - 3$ because $2\left(-\frac{1}{2}\right)^2 - 5\left(-\frac{1}{2}\right) - 3 = 0$.

Theorem: A polynomial of degree n cannot have more than n different roots.

ADDING AND SUBTRACTING POLYNOMIALS

Only like terms can be added or subtracted together into one term: **Ex:** $3x^3y - 5x^3y = -2x^3y$.

MULTIPLYING POLYNOMIALS

Multiply every term of the first by every term of the second, term by term. The number of terms in the (unsimplified) product is the product of the numbers of terms in the two polynomials.

Multiplying a monomial by any other polynomial: Distribute and multiply each term of the polynomial by the monomial.

Multiplying two binomials: Multiply each term of the first by each term of the second: $(a+b)(c+d) = ac + ad + bc + bd$.

FOIL (mnemonic): Multiply the two **F**irst terms, the two **O**utside terms, the two **I**nside terms, and the two **L**ast terms.

Common products:
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)(a-b) = a^2 - b^2$$

After multiplying, simplify by combining like terms.

FACTORING POLYNOMIALS

Like numbers, polynomials are built of "primes"—polynomials that cannot be factored. We will assume that all coefficients are whole numbers.

- Factoring polynomials is the reverse of applying the distributive property. Reexpressing $(ab + ac)$ as $a(b+c)$ is called **factoring out** a.

FACTORING OUT COMMON FACTORS

1. Factor out the **Greatest Common Factor (GCF)** of all the coefficients in the polynomial.
2. If any variable appears in every term, factor out the least power of that variable that occurs in the polynomial.

$x^2 + bx + c$: FACTORING TRINOMIALS WITH A LEADING COEFFICIENT OF ONE

If $x^2 + bx + c$ factors (which is not guaranteed), it will factor as $(x+d)(x+e)$. The goal is to find two numbers d and e whose sum is b and whose product is c.

1. Find all possible pairs of integers whose product is c.
2. For each pair, test whether its sum is b.

If b, c, d, and e are all positive...

$x^2 + bx + c = (x+d)(x+e)$.
Check if any (positive) factor pairs of c sum to b.

$x^2 - bx + c = (x-d)(x-e)$.
Check if any (positive) factors pairs of c sum to b.

$x^2 \pm bx - c = (x-d)(x+e)$.
Check if the difference of any (positive) factors pairs is b. This is usually the trickiest case.

$ax^2 + bx + c$: FACTORING GENERAL TRINOMIALS

The trinomial $ax^2 + bx + c$ will factor as $(a_1x + b_1)(a_2x + b_2)$.

1. Compute ac. Find all possible pairs of integers (positive and negative) whose product is ac.
2. For each pair of integers, test if its sum is b.
3. Suppose you find a pair with $d_1d_2 = ac$ and $d_1 + d_2 = b$. Rewrite the equation as $(ax^2 + d_1x) + (d_2x + c)$.
4. Factor out the Greatest Common Factor (from both the variables and the coefficients) for the two expressions in parentheses individually. The two remains of factoring should look the same:
$$a_1x(a_2x + b_2) + b_1(a_2x + b_2).$$
5. Finally, rewrite as $(a_1x + b_1)(a_2x + b_2)$.

Ex: Factor $6x^2 - x - 12$.

Look for two numbers whose product is $6(-12) = -72$ and whose sum is -1. Guessing and checking, find that 8 and -9 work.
Reexpress:
$$6x^2 - x - 12 = (6x^2 + 8x) + (-9x - 12).$$
Factoring out the GCF from each group, obtain
$$2x(3x+4) - 3(3x+4) = (2x-3)(3x+4).$$

If we had written 8 and -9 in reverse order, we would have gotten the same result:
$$(6x^2 - 9x) + (8x - 12) =$$
$$3x(2x-3) + 4(2x-3) = (3x+4)(2x-3).$$

RECOGNIZING EASY-TO-FACTOR GROUPS

Perfect Squares:	$a^2 + 2ab + b^2 = (a+b)^2$ $a^2 - 2ab + b^2 = (a-b)^2$
Difference of Squares:	$a^2 - b^2 = (a+b)(a-b)$
Sum & Difference of Cubes:	$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$ $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

ADVANCED TECHNIQUES

Sometimes, a difference of squares is hiding.
Ex: $x^2 - y^2 - 4x + 4$ can be factored by grouping all the x-terms together and factoring those separately first:
$(x^2 - 4x + 4) - y^2 = (x-2)^2 - y^2 = (x+y-2)(x-y-2)$.

A sum of squares usually can't be factored. But it will if the square is really a higher power. **Ex:** $a^6 + b^6$ will factor as a sum of cubes.
Ex (very hard): $a^4 + 4b^4 = (a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2$
$= (a^2 + 2b^2)^2 - (2ab)^2 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$.

Any difference of powers can be factored:
$$a^5 - b^5 = (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4);$$
$$a^6 - b^6 = (a-b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5);\text{ etc.}$$

Any sum of *odd* powers can be factored:
$$a^5 + b^5 = (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4),\text{ etc.}$$

When dealing with composite powers, treat them as a smallest-factor power first. **Ex:** $a^{15} - b^{15}$ should be factored as a difference of cubes first:
$(a^5)^3 - (b^5)^3 = (a^5 - b^5)(a^{10} + a^5b^5 + b^{10})$;
then factor the $a^5 - b^5$ piece separately.

QUADRATIC EQUATIONS IN ONE VARIABLE

A **quadratic equation in one variable** (say, x) is an equation that can be converted into the form $ax^2 + bx + c = 0$, where a, b, and c are real constants.

A **solution** to the equation (a value of x that makes the equation true) is always a root of the polynomial $ax^2 + bx + c$.

LOOKING FOR ROOTS IS FACTORING

The **roots** of the quadratic $(x-c)(x-d)$ are exactly c and d.

True because of the Zero Product Property: $ab = 0$ if and only if $a = 0$ or $b = 0$. So $(x-c)(x-d) = 0$ if and only if $x - c = 0$ or $x - d = 0$; equivalently if $x = c$ or $x = d$.

Similarly, the roots of the quadratic $(ax+b)(cx+d)$ are $-\frac{b}{a}$ and $-\frac{d}{c}$.

To find the solutions to a quadratic equation in the form $ax^2 + bx + c = 0$ try factoring the polynomial first. See *Factoring Polynomials,* above.

COMPLETING THE SQUARE

Completing the square is a procedure for solving a quadratic equation that is difficult or impossible to factor.

Ex: Finding solutions to $x^2 + 5x + 6 = 0$.

1. Isolate the x terms on one side. **Ex:** $x^2 + 5x = -6$
2. Add something to both sides that would make the x side a perfect square. If the x^2 coefficient is 1, add the square of half the x coefficient. **Ex:** $x^2 + 5x + \left(\frac{5}{2}\right)^2 = -6 + \left(\frac{5}{2}\right)^2$ simplifies to $x^2 + 5x + \frac{25}{4} = \frac{1}{4}$.
3. Take the square root of both sides. If the constant side is negative, there are no solutions to the equation; if positive, there are two solutions; if 0, there is exactly one solution to the equation. **Ex:** $\left(x + \frac{5}{2}\right)^2 = \left(\frac{1}{2}\right)^2$. Therefore $x + \frac{5}{2} = \pm\frac{1}{2}$.
4. Find the solutions. **Ex:** $x + \frac{5}{2} = \frac{1}{2}$ or $x + \frac{5}{2} = -\frac{1}{2}$ simplifies to $x = -2$ or $x = -3$.

Alternatively, we could have realized that $x^2 + 5x + 6$ factors as $(x+2)(x+3)$; the solutions must therefore be $x = -2, -3$.

QUADRATIC FORMULA

Completing the square on a general quadratic gives the...

Quadratic Formula: The solutions to $ax^2 + bx + c = 0$ are given by:
$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The expression under the square root $b^2 - 4ac$ is called the **discriminant;** it determines how many solutions the quadratic equation $ax^2 + bx + c$ has:

- If $b^2 - 4ac > 0$ then there are two distinct real solutions.
- If $b^2 - 4ac = 0$ then there is one real solution (and $ax^2 + bx + c$ is a perfect square).
- If $b^2 - 4ac < 0$ then there are no real solutions.

SUM AND PRODUCT OF ROOTS

For a polynomial $ax^2 + bx + c$...

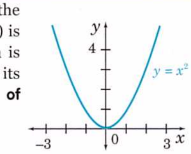
The **sum** of its two roots is $-\frac{b}{a}$.

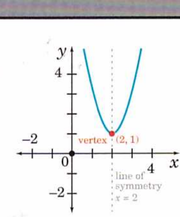
The **product** of its two roots is $\frac{c}{a}$.

This is true even if the roots are not real. See *Complex Numbers.*

GRAPHING QUADRATIC EQUATIONS

The graph of every equation of the form $y = ax^2 + bx + c$ (with $a \neq 0$) is a **parabola**. The simplest parabola is the graph of $y = x^2$: it opens up, its **vertex** is at the origin, and its **line of symmetry** is the vertical line $x = 0$.





$y = 2(x-2)^2 + 1$

The general parabola:
$$y = ax^2 + bx + c.$$
Complete the square to reexpress the equation in the easy-to-graph form $y = a(x-h)^2 + k$. Here, $h = -\frac{b}{2a}$ and k is the value obtained when h is plugged into the polynomial $ax^2 + bx + c$.

The **vertex** of the parabola is at (h, k).

The line of symmetry is $x = -\frac{b}{2a}$.

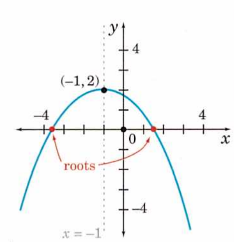
The parabola opens up if $a > 0$, down if $a < 0$.

If $|a| > 1$, the parabola is steeper than $y = x^2$; if $|a| < 1$, the parabola opens wider than $y = x^2$.

The roots of the polynomials are the x-values where the graph crosses the x-axis.

The sign of the discriminant $b^2 - 4ac$ determines the number of roots. See *Quadratic Formula.*

If real roots exist, the line of symmetry parses exactly half-way between them (so the x-coordinate of the line of symmetry is the average of the roots).



$y = -\frac{1}{3}(x+1)^2 + 2$

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ALGEBRA II

POLYNOMIAL DIVISION

We can use **long division** to divide one polynomial in one variable by another (possibly with a remainder) much like we do with integers.

Terminology flashback: In the expression "$50 \div 3 = 16$, remainder 2," 50 is the **dividend,** 3 is the **divisor,** 16 is the **quotient,** and 2 is the **remainder.** Using integers, we can always divide leaving a remainder strictly less than the (absolute value of the) divisor.

In polynomial division, the remainder must be 0 or of a smaller *degree* than the divisor.

Instead of digit by digit, polynomial division proceeds term by term:

1. Write both polynomials in order of descending degree; it may be helpful to insert $0x^n$ for every term with coefficient 0.
2. Divide the leading term of the dividend by the leading term of the divisor to get the first term of the quotient (the coefficient may not be an integer).
3. Multiply the quotient term by the divisor and subtract the product from the dividend; the difference should have smaller degree than the original dividend.
4. Repeat, using the difference as the new dividend, until the next "new dividend" is 0 (the divisor is a factor of the dividend) or the new dividend has degree strictly smaller than the degree of the divisor (this last new dividend is the remainder).

Ex:

$$\begin{array}{rl} & x^2 - \tfrac{1}{2}x + 3 \\ 2x^2 - 4x + 3 \,\big)\!\! & \overline{2x^4 - 5x^3 + 11x^2 - 9x + 5} \\ & \underline{-(2x^4 - 4x^3 + 3x^2)} \\ & \qquad -x^3 + 8x^2 - 9x + 5 \\ & \qquad \underline{-(-x^3 + 2x^2 - \tfrac{3}{2}x)} \\ & \qquad\qquad 6x^2 - \tfrac{15}{2}x + 5 \\ & \qquad\qquad \underline{-(6x^2 - 12x + 9)} \\ & \qquad \text{remainder: } \tfrac{9}{2}x - 4 \end{array}$$

SYNTHETIC DIVISION

Synthetic Division is a faster, slightly trickier way of dividing a polynomial by a binomial of the form $x - a$.

To divide by a binomial of the form $ax + b$, use synthetic division to divide by $x - \left(-\frac{b}{a}\right)$, then divide the result by a.

There are three lines:

1. In line 1, write the potential root (a if dividing by $x - a$). To the right on the same line, write the coefficients of the polynomials in descending order of degree, including a 0 coefficient for all missing terms (even the constant).
2. Leaving space for line 2, draw a horizontal line under the coefficients. Copy the leading coefficient into line 3 under the horizonal line.
3. Multiply that entry in line 3 by a and write the result in line 2, under the second coefficient. The first position of line 2 is blank.
4. Add the numbers in the second position of lines 1 and 2; write the sum in line 3.
5. Repeat—multiply the new entry of line 3 by a, write in next position in line 2; add entries in lines 1 and 2, write in line 3—until done.
6. The last entry in line 3 is the remainder. The rest of line 3 represents the coefficients of the quotient, in descending order of degree. The degree of the quotient is one less than the degree of the dividend.

Ex:

$$\begin{array}{r|rrrrr} -2 & 2 & 4 & -3 & -1 & 3 \\ & & -4 & 0 & 6 & -10 \\ \hline & 2 & 0 & -3 & 5 & \boxed{-7} \text{ remainder} \end{array}$$

Synthetic division:
$(2x^4 + 4x^3 - 3x^2 - x + 3) \div (x + 2)$
$= 2x^3 - 3x + 5$, remainder -7.

CONSEQUENCES OF POLYNOMIAL DIVISION

Remainder Theorem: If $f(x)$ is a polynomial, then the remainder from dividing $f(x)$ by $x - a$ is the value $f(a)$.

True because if $f(x) = (x - a)q(x) + r$, where $q(x)$ is the quotient and r is the (constant) remainder, then plugging in a gives

$$f(a) = 0 \cdot q(a) + r.$$

Factor Theorem: a is a root of the polynomial $f(x)$ if and only if $x - a$ is a factor of $f(x)$.

Bottom line: When looking for roots, try factoring. When trying to determine whether $ax + b$ is a factor, plug in $-\frac{b}{a}$.

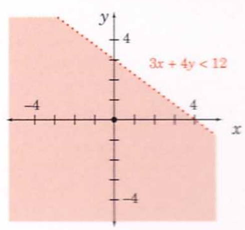
INEQUALITIES IN TWO VARIABLES

Graphing transforms the problem of finding solutions to inequalities into identifying a **region of solutions** in the Cartesian plane. Usually, the region of solutions is shaded in a particular way. A **solid boundary line** (——) indicates that the points on the boundary are also solutions. A **dashed boundary line** (- - - -) means that the boundary points are not solutions.

GRAPHING A SINGLE INEQUALITY

A single inequality will give a **half plane** of solutions.

1. Graph the corresponding equality. $>$ and $<$ signs give dashed lines; $\geq$ and $\leq$ signs give solid lines.
2. If the equation is in $y \lesseqgtr mx + b$ form: $>$ or $\geq$ sign means that the shaded solution region is above the line; $<$ or $\leq$ sign means that the shaded solution region is below the line.
   - Alternatively, just test a point not on the line in the inequality to determine which side is the solution half-plane.



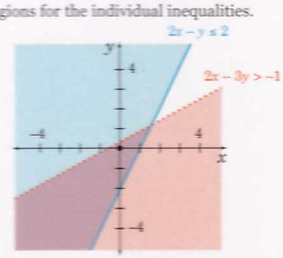
Shaded region: $3x + 4y < 12$

- To graph $y \neq mx + b$, shade everything but the (dashed) line.

GRAPHING SYSTEMS OF INEQUALITIES

Several inequalities will give a **region of solutions** in the plane. The region may be finite or infinite; it may be a sector, a strip, or a polygon.

- Graph all of the inequalities in the system individually. The lines of the graphs will divide the coordinate system into 2, 4, or more sections. Check a point in each section for each inequality.
- If the inequalities are joined by AND, the solution region is the intersection of solution regions for the individual inequalities.
- If the inequalities are joined by OR, the solution region is the union of solution regions for the individual inequalities.



Shaded region: $2x - y \leq 2$ OR $2x - 3y > -1$.
Purple region: $2x - y \leq 2$ AND $2x - 3y > -1$.

Optimization questions reduce to the problem of maximizing or minimizing some variable expression, subject to some condition on the variables (usually some inequality).

- Likely suspects are word problems that ask for the best possible use of resources based on cost or capacity of a person or organization.

To solve, graph the inequalities. The inequalities will define (sometimes in conjunction with the coordinate axes) a region of feasible solutions.

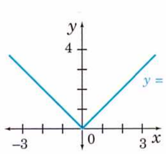
- If the region is finite, calculate the value of the expression being maximized or minimized at *each corner of the region.* If the expression is linear and there is a unique solution, it will always be a corner (or an apex of a curve).
- If the region is infinite, also check some points in the unbounded part to make sure that maxima or minima exist.

GRAPHING ABSOLUTE VALUE

The **absolute value** of a is the distance from a to 0. Formally,

$$|a| = \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}.$$

The graph of the equation $y = |x|$ has a $\vee$ shape:



The graph of the equation $y = a|x - h| + k$ behaves just like the graph of the quadratic $y = a(x - h)^2 + k$:

- The vertex is at $(h,\ k)$.
- The graph opens up if $a > 0$, down if $a < 0$.
- The graph is steep if $|a| > 1$, gently sloping if $|a| < 1$.
- The x-axis taken alone is the graph of the single-variable equation $a|x - h| + k = 0$ on the real-number line.

LOGARITHMS

The expression $\log_a b$ is read "logarithm (or log) base a of b."

Logarithms are exponents: $\log_a b$ is "the power to which you raise a to get b":

$\log_a b = n$ means that $a^n = b$.

- a is the **base.** It is always a positive number not equal to 1 (often, $a > 1$).
- b must also be positive.

LOGARITHM LAWS

Restating definitions:

$\log_a(a^n) = n$ and $a^{\log_a b} = b$

Log of 1:	$\log_a 1 = 0$
Log of the base:	$\log_a a = 1$
Log of product:	$\log_a(bc) = \log_a b + \log_a c$
Log of quotient:	$\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$
Log of reciprocal:	$\log_a \frac{1}{b} = -\log_a b$
Log of power:	$\log_a b^n = n \log_a b$
Log of root:	$\log_a \sqrt[n]{b} = \frac{1}{n} \log_a b$
Log of fractional exponent:	$\log_a \sqrt[n]{b^m} = \frac{m}{n} \log_a b$
Change of base:	$\log_a b = \frac{\log_c b}{\log_c a}$
	$\log_a b = \frac{1}{\log_b a}$

LOGARITHMS BASE 10

Log base 10 is often used in applied sciences (and by calculators). It is sometimes called the **common logarithm.**

Notation: Drop the base subscript: $\log b$ is understood to mean $\log_{10} b$.

Converting logs to base 10: $\log_a b = \frac{\log b}{\log a}$.

LOGARITHMS BASE e

The real number e is a special irrational number (approximately 2.71828) often used as a base for logarithms. The logarithm base e is called the **natural logarithm.**

The natural log follows **all** logarithm rules.

Notation: $\log_e x$ is written as $\ln x$.

Converting: Any logarithmic expression can be written in terms of natural logarithms using the change of base formula

$$\log_a b = \frac{\ln b}{\ln a}.$$

$a^x = b$: EXPONENTIAL EQUATIONS

To solve: Take logarithms of both sides if everything is positive: if $b = c$ and both are positive, then $\log_a b = \log_a c$ for any base $a > 0$.

Ex: $a^x = b$ means $\log_a a^x = \log_a b$. So $x = \log_a b$.

- To evaluate x with a calculator, compute $x = \frac{\log b}{\log a} = \frac{\ln b}{\ln a}$.

SEQUENCES AND SERIES

SEQUENCES: DEFINITIONS

A **sequence** is an ordered list of real numbers, called **terms.** An **infinite** sequence has infinitely many terms.

Notation: $\{a_k\}_{k=1}^{\infty}$ often represents the sequence $a_1, a_2, a_3, \ldots$

- Sequences can be **defined explicitly** by giving a formula for the n^{th} term:

Ex: $a_n = n^2$ is the sequence $1, 4, 9, 16, \ldots$

- Or **defined recursively** by defining the n^{th} in terms of preceding terms:

Ex: $a_1 = 1$; $a_n = a_{n-1} + (2n - 1)$ is again the sequence $1, 4, 9, \ldots$

SERIES: DEFINITIONS

A **series** is a summed sequence: $a_1 + a_2 + a_3 + \cdots$. An **infinite series** has infinitely many terms.

- **Sigma notation:** A finite series is sometimes written as

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k.$$

An infinite series can be written as

$$a_1 + a_2 + \cdots = \sum_{k=1}^{\infty} a_k.$$

- A **partial sum** of a series is a cut-off series sum:

$$s_n = a_1 + a_2 + \ldots + a_n = \sum_{k=1}^{n} a_k.$$

CONTINUED ON OTHER SIDE

SEQUENCES AND SERIES (CONTINUED)

- The **sum** of a series, if it exists, is the value that the partial sums get very close to as n gets large. If the sum exists, we say the sequence **converges**; otherwise it **diverges**.

Ex 1: The series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ sums to 1:

Ex 2: The series $1-1+1-1+\cdots$ has no sum: the partial values alternate between 0 and 1 and do not approach anything.

ARITHMETIC SEQUENCES AND SERIES

An **arithmetic sequence** is a sequence whose successive terms differ by a constant, called the **common difference.** It is defined by its first term and the common difference (often denoted by d). It has the form

$$a,\ a+d,\ a+2d,\ a+3d,\ldots$$

Ex: $3, 10, 17, 24, \ldots$ is an arithmetic sequence with first term 3 and common difference 7.

- The n^{th} term of the sequence with first term a_1 and common difference d is

$$a_n = a_1 + (n-1)d.$$

- The n^{th} partial sum of an **arithmetic series** is $s_n = \frac{1}{2}n(a_n + a_1) = \frac{1}{2}n(2a_1 + (n-1)d)$.
- An infinite arithmetic sequence never converges.

GEOMETRIC SEQUENCES AND SERIES

A **geometric sequence** is a sequence whose successive terms are multiplied by a constant, called the **common ratio.** It is defined by the first term and the common ratio (often r):

$$a,\ ar,\ ar^2,\ ar^3,\ldots$$

Ex: $1, -3, 9, -27, \ldots$ is a geometric series with first term 1, and ratio -3.

- The n^{th} term of the sequence with first term a_1 and common ratio r is $a_n = a_1 r^{n-1}$.
- The n^{th} partial sum of a **geometric series** is

$$s_n = \frac{a_{n+1}-a_1}{r-1} = a_1\frac{r^n-1}{r-1}.$$

The formula holds only if $r \neq 1$ (otherwise the partial sum is na_1).

- An infinite geometric sequence converges if and only if $|r| < 1$, in which case the sum is $s = \frac{a_1}{1-r}$.

COUNTING: DISCRETE FUNCTIONS

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1
1 + 1
1 + 2 + 1
1 + 3 + 3 + 1
1 + 4 + 6 + 4 + 1
1 5 10 10 5 1

Pascal's Triangle (rows 0–5)

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$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + \binom{n}{n-1}ab^{n-1} + b^n$$

We can read off the coefficients of the powers of $a+b$ from Pascal's Triangle.

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$$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$$
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

We can use the theorem to find a power of any binomial.

Ex: $(2x-3)^3 = (2x)^3 + 3(2x)^2(-3) + 3(2x)(-3)^2 + (-3)^3 = 8x^3 - 36x^2 + 54x - 27$

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IMAGINAR

- **Imaginary** nu
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- Use $\sqrt{-a} =$ reals and i n
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- All powers o number.

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Complex numbe Complex numbe numbers.

- **Conjugation:** $\overline{a+bi} = a -$
- $(a+bi)(a-$

CONI

A **conic section** dimensional con section is th $Ax^2 + By^2 + C$ case $C = 0$.

- To graph a co on the x and square on A $By^2 + Ey$. T $A(x-h)^2 +$ manipulated i

PARABOLA

- Basic parabol
- General paral $y = a(x-h)^2$ $k = c - ah^2$.
- The **vertex** is a
- The parabola
- a determines (when $|a| < 1$ gives a narrow
- **Set definition:** equidistant fr line called the
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- A conic sectio term is 0 (but
- A parabola is to the cone's s

Parabola w

CIRCLE

- Basic circle: x radius 1.
- General circle: $(x-h)^2 + (y$
- r is the **radius.**
- (h,k) is the **ce**
- **Set definition:** all points a dis away from (h,k)—the cer
- A conic section term and the same coefficier
- A circle is perpendicular

SEQUENCES AND SERIES (CONTINUED)

- The **sum** of a series, if it exists, is the value that the partial sums get very close to as n gets large. If the sum exists, we say the sequence **converges;** otherwise it **diverges.**

Ex 1: The series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ sums to 1:
Ex 2: The series $1-1+1-1+\cdots$ has no sum: the partial values alternate between 0 and 1 and do not approach anything.

ARITHMETIC SEQUENCES AND SERIES

An **arithmetic sequence** is a sequence whose successive terms differ by a constant, called the **common difference.** It is defined by its first term and the common difference (often denoted by d). It has the form

$$a,\ a+d,\ a+2d,\ a+3d, \ldots$$

Ex: $3, 10, 17, 24, \ldots$ is an arithmetic sequence with first term 3 and common difference 7.

- The n^{th} term of the sequence with first term a_1 and common difference d is

$$a_n = a_1 + (n-1)d.$$

- The n^{th} partial sum of an **arithmetic series** is $s_n = \frac{1}{2}n(a_n + a_1) = \frac{1}{2}n(2a_1 + (n-1)d)$.
- An infinite arithmetic sequence never converges.

GEOMETRIC SEQUENCES AND SERIES

A **geometric sequence** is a sequence whose successive terms are multiplied by a constant, called the **common ratio.** It is defined by the first term and the common ratio (often r):

$$a,\ ar,\ ar^2,\ ar^3, \ldots$$

Ex: $1, -3, 9, -27, \ldots$ is a geometric series with first term 1, and ratio -3.

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$$s_n = \frac{a_{n+1} - a_1}{r-1} = a_1 \frac{r^n - 1}{r-1}.$$

The formula holds only if $r \neq 1$ (otherwise the partial sum is na_1).

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```
            1
          1   1
        1   2   1
      1   3   3   1
    1   4   6   4   1
  1   5  10  10   5   1
```

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COMPLEX NUMBERS

The set of **complex** numbers includes the set of real numbers. Every polynomial of degree n has exactly n complex number roots (counting multiple roots).

IMAGINARY NUMBERS

- **Imaginary** numbers are square roots of negative numbers.
- $i = \sqrt{-1}$ is the most basic imaginary number; every other imaginary number can be expressed in terms of i.
- Use $\sqrt{-a} = i\sqrt{a}$ to convert between square roots of negative reals and i notation.
- Powers of i : $i^1 = i$; $i^2 = -1$; $i^3 = -i$; $i^4 = 1$. Also, $\frac{1}{i} = i^3 = -i$, and $i^n = i^r$ where r is the remainder that n leaves when divided by 4.
- All powers of imaginary numbers reduce to i times a real number.

COMPLEX NUMBERS

Complex numbers are all numbers $a + bi$ where a and b are real. Complex numbers are all sums and products of real and imaginary numbers.

- **Conjugation**: The **complex conjugate** of $a + bi$ is $\overline{a+bi} = a - bi$. Also, $\overline{a - bi} = \overline{a + \overline{bi}} = a + bi$.
- $(a + bi)(a - bi) = a^2 + b^2$, a real number and a sum of 2 squares.
- **Addition** and **subtraction:** Complex numbers are added like polynomials, keeping the real and the imaginary parts separate. So $(a + bi) + (c + di) = (a + c) + (b + d)i$.
- **Multiplication:** Use $i \cdot i = -1$. So $(a + bi)(c + di) = (ac - bd) + (bc + ad)i$.
- When computing powers, we can think of multiplying polynomials with i as the variable. Afterward, reduce powers of i as above and add like terms.
- **Division:** Reexpress $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$. This is the complex number $\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$.
- To divide one complex number by another, multiply top and bottom of the fraction by the conjugate of the denominator and simplify the numerator. Thus $\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}$.
- **Rationalizing the denominator:** "Simplified form" for a complex fraction means having no i in the denominator. Get rid of them by multiplying top and bottom by the conjugate of the denominator.

COMPLEX NUMBERS AND THE QUADRATIC FORMULA

The quadratic $ax^2 + bx + c$ has two real roots if the discriminant $b^2 - 4ac$ is positive. If the discriminant is zero, it has one **double root.** If the discriminant is negative, it also has two roots, but this time the roots are complex.

- The **Quadratic Formula** still works; the two roots are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$.
- The sum and product of roots formulas also work: Sum is $-\frac{b}{a}$; product is $\frac{c}{a}$.

FUNDAMENTAL THEOREM OF ALGEBRA

Every polynomial factors completely if complex roots are allowed. Counting multiple roots, it has exactly as many roots as its degree.

THE COMPLEX PLANE

If real numbers are represented as points on a line, complex numbers can be represented as points on the **complex plane:** the number $a + bi$ is the point $(a,\ b)$. By convention, the x-axis is the real axis, and the y-axis is the imaginary axis.

- Addition works like addition of vectors in the plane.
- Multiplying $a + bi$ by a real number stretches the vector $(a,\ b)$ radially from the origin.
- Multiplying $a + bi$ by -1 flips the vector $(a,\ b)$ through the origin.
- Multiplying $a + bi$ by i rotates $(a,\ b)$ by $90°$ counter-clockwise. Multiplying by $-i$ rotates by $90°$ clockwise.

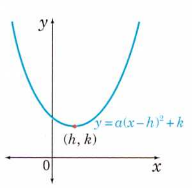
CONIC SECTIONS

A **conic section** is the shape created by the intersection of a 3-dimensional cone and a plane cutting through it. A general conic section is the set of all solutions to the relation $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$. We only work with the case $C = 0$.

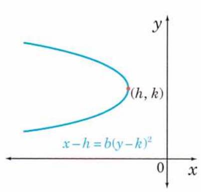
- To graph a conic section, it is easiest to complete the square on the x and y terms separately: if $A \neq 0$, complete the square on $Ax^2 + Dx$; if $B \neq 0$, complete the square on $By^2 + Ey$. The goal is to create an expression like $A(x - h)^2 + B(y - k)^2 = G$, which can then be manipulated into an easy-to-graph form.

PARABOLA

- Basic parabolas: $y = x^2$ opens up; $x = y^2$ opens to the right.
- General parabola: $y = ax^2 + bx + c$. Rewrite in the form $y = a(x - h)^2 + k$ by completing the square: $h = -\frac{b}{2a}$ and $k = c - ah^2$.
- The **vertex** is at (h, k). This is the turning point.
- The parabola opens up if $a > 0$; down if $a < 0$.
- a determines how wide or narrow the parabola is. Small a (when $|a| < 1$) gives a wide parabola. Large a (when $|a| > 1$) gives a narrow parabola.
- **Set definition:** A parabola is the set of points in the plane equidistant from a special point called the **focus** and a special line called the **directrix.**
- The directrix is the line $y = h - \frac{1}{4a}$.
- A conic section is a parabola if either the y^2 term or the x^2 term is 0 (but not both).
- A parabola is the intersection of a cone with a plane parallel to the cone's slant.



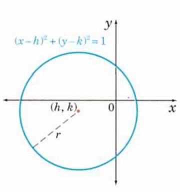
Parabola with $a > 0$



Parabola with $b < 0$

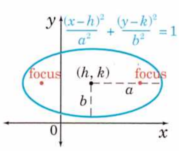
CIRCLE

- Basic circle: $x^2 + y^2 = 1$ is centered at the origin and has radius 1.
- General circle: $(x - h)^2 + (y - k)^2 = r^2$.
- r is the **radius.**
- (h, k) is the **center** of the circle.
- **Set definition:** A circle is the set of all points a distance r, the radius, away from a specific point (h, k)—the center of the circle.
- A conic section is a circle if the x^2 term and the y^2 term have the same coefficient (including the same $\pm$ sign).
- A circle is the intersection of a cone with a plane perpendicular to its main axis.



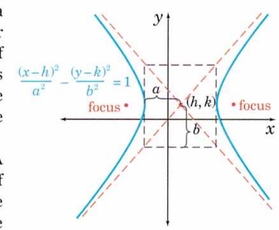
ELLIPSE

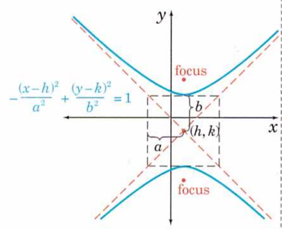
- Basic ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is centered at the origin. The **semi-major** and **semi-minor** axes have lengths a and b (the larger one is major, the smaller is minor).
- General ellipse: $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.
- a and b are the lengths of the half-axes. If $a > b$, then the ellipse is elongated left to right; if $a < b$, the ellipse is elongated up and down.
- (h, k) is the center of the ellipse.
- **Set definition:** An ellipse is the set of points the sum of whose distances from two fixed points, called the **foci,** is constant.
- The foci of an ellipse lie along its major axis, inside the ellipse, equidistant from the center. The **focal distance** (distance from the foci to the center) is given by $c = \sqrt{|a^2 - b^2|}$. If $a > b$, then the foci are at $(h + c, k)$ and $(h - c, k)$. If $b > a$, then the foci are at $(h, k \pm c)$.
- The constant distance sum is the length of the full major axis: $2a$ or $2b$, whichever is larger.
- A conic section is an ellipse if neither the x^2 term nor the y^2 term is zero and if they both have the same sign.
- If the x^2 term and the y^2 term are the same, then the ellipse is degenerate; it is actually a circle. The focal distance is 0.
- An ellipse is the intersection of a cone with a plane at an angle greater than the cone's slant angle.

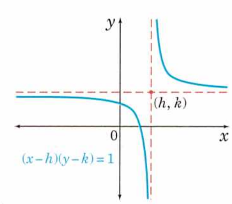


HYPERBOLA

- Basic hyperbola: $x^2 - y^2 = \pm 1$ is centered at the origin. Lines $y = \pm x$ are asymptotes. The hyperbola with the + sign: $x^2 - y^2 = 1$ opens left-right (x-direction); $y^2 - x^2 = 1$ opens up-down (y-direction).
- More general hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$.
- If the right side of equation is $+1$, hyperbola opens left-right. The vertices are the points $(a, 0)$ and $(-a, 0)$.
- If the right side of the equation is -1, hyperbola opens up-down. The vertices are the points $(0, \pm b)$.
- Diagonal lines $y = \pm\frac{b}{a}$ are asymptotes.
- Most general hyperbola: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = \pm 1$ has center at (h, k).
- **Drawing a hyperbola:** Find the center (h, k). Plot the $2a$-by-$2b$ box around it; the four corners are the points $(h \pm a, k \pm b)$. Draw in the diagonals to the box and extend them out as lines $y = \pm\frac{b}{a}(x - h) + k$. Depending on which way the hyperbola opens, the up-down or left-right midpoints of the box are the vertices of the hyperbola; the diagonal lines are asymptotes.
- **Set definition:** A hyperbola is the set of all points the difference of whose distances from two fixed points, called the **foci,** is constant.
- The foci of an ellipse lie inside the curve of the two halves of the hyperbola: along the left–right or up–down axis (depending on which way the hyperbola opens), equidistant from the center. The focal distance, c, is given by $c = \sqrt{a^2 + b^2}$.
- The constant difference is $2a$ if the hyperbola opens left-right; $2b$ if it opens up–down.
- A conic section is a hyperbola if the x^2 and the y^2 terms (both nonzero) have different signs.
- If, after completing the square, there is no constant term left, then the hyperbola is **degenerate:** the equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ and the graph is the two diagonal lines $y = \pm\frac{b}{a}$.
- A conic section is also a hyperbola if there is a nonzero xy term while the x^2 and y^2 terms are zero. Such a hyperbola has an equation like $xy = 1$ and the two axes are its asymptotes.
- A hyperbola is the intersection of a cone with a plane at an angle less than the cone's slant angle.







LINE

- A straight line is also a conic section—with the x^2, y^2, and xy terms all zero.

WHICH CONIC SECTION IS WHICH?

General conic $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$. Assume $C = 0$ everywhere but in the last entry.

If the coefficients...	...then the conic section is a(n)...
$A = B = 0$	Line. Line is vertical if $E = 0$. Line is horizontal if $D = 0$.
$A \neq 0, B = 0$	Parabola opening up (if A and E have opposite signs) or down (if A and E have the same sign).
$A = 0, B \neq 0$	Parabola opening right (if B and D have opposite signs) or left (if B and D have the same sign).
$A = B \neq 0$	Circle.
$A, B > 0$ or $A, B < 0$	Ellipse oriented left-right (if $\lvert A\rvert < \lvert B\rvert$) or up-down (if $\lvert A\rvert > \lvert B\rvert$.)
$A < 0, B > 0$ or $A > 0, B < 0$	Hyperbola opening left–right or up–down. (Complete the square to get $A(x - h)^2 + B(y - k)^2 = H$; it opens left–right if $\frac{H}{A} > 0$, up–down if $\frac{H}{A} < 0$, and is degenerate if $H = 0$.)
$A = B = 0, C \neq 0$	Hyperbola with a vertical and a horizontal asymptote. (Factor to get form $C(x - h)(y - k) = H$; opens to quadrants I and III if $\frac{H}{C} > 0$, to quadrants II and IV if $\frac{H}{C} < 0$, and is degenerate if $H = 0$).

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